Exercise 37

The displacement (in meters) of an object moving in a straight line is given by $s = 1 + 2t + \frac{1}{4}t^2$, where t is measured in seconds.

(a) Find the average velocity over each time period.

(i) $[1, 3]$ (ii) $[1, 2]$ (iii) $[1, 1.5]$ (iv) $[1, 1.1]$

(b) Find the instantaneous velocity when $t = 1$.

Solution

Part (a)

The average velocity over each time period is given by the slope of the secant line.

(i) $[1, 3]$: $$m = \frac{s(3) - s(1)}{3 - 1} = \frac{\left[1 + 2(3) + \frac{1}{4}(3)^2\right] - \left[1 + 2(1) + \frac{1}{4}(1)^2\right]}{2} = 3$$

(ii) $[1, 2]$: $$m = \frac{s(2) - s(1)}{2 - 1} = \frac{\left[1 + 2(2) + \frac{1}{4}(2)^2\right] - \left[1 + 2(1) + \frac{1}{4}(1)^2\right]}{1} = 2.75$$

(iii) $[1, 1.5]$: $$m = \frac{s(1.5) - s(1)}{1.5 - 1} = \frac{\left[1 + 2(1.5) + \frac{1}{4}(1.5)^2\right] - \left[1 + 2(1) + \frac{1}{4}(1)^2\right]}{0.5} = 2.625$$

(iv) $[1, 1.1]$: $$m = \frac{s(1.1) - s(1)}{1.1 - 1} = \frac{\left[1 + 2(1.1) + \frac{1}{4}(1.1)^2\right] - \left[1 + 2(1) + \frac{1}{4}(1)^2\right]}{0.1} = 2.525$$

The units of these average velocities are meters per second.

Part (b)

To find the instantaneous velocity when $t = 1$, calculate the derivative of $s(t)$ and then set $t = 1$. Use the definition of the derivative.

$$\begin{aligned}
s'(t) &= \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} \\
&= \lim_{h \to 0} \frac{\left[1 + 2(t+h) + \frac{1}{4}(t+h)^2\right] - \left(1 + 2t + \frac{1}{4}t^2\right)}{h} \\
&= \lim_{h \to 0} \frac{\left[1 + 2(t+h) + \frac{1}{4}(t^2 + 2th + h^2)\right] - \left(1 + 2t + \frac{1}{4}t^2\right)}{h} \\
&= \lim_{h \to 0} \frac{\left(1 + 2t + 2h + \frac{1}{4}t^2 + \frac{1}{2}th + \frac{1}{4}h^2\right) - 1 - 2t - \frac{1}{4}t^2}{h} \\
&= \lim_{h \to 0} \frac{2h + \frac{1}{2}th + \frac{1}{4}h^2}{h}
\end{aligned}$$

Cancel out h and evaluate the limit.

$$s'(t) = \lim_{h \to 0} \left(2 + \frac{1}{2}t + \frac{1}{4}h\right)$$

$$= 2 + \frac{1}{2}t$$

The instantaneous velocity when $t = 1$ is then

$$s'(1) = 2 + \frac{1}{2}(1) = 2.5,$$

where the units are in meters per second.